

Maximal irreducibility measure for spatial birth-and-death processes

Viktor Bezborodov ^{*}and Luca Di Persio [†]

University of Verona - Department of Computer Science

Keywords: birth-and-death process; Maximal irreducibility measure; Markov chain; the Lebesgue-Poisson measure, ϕ -irreducibility

AMS classification: 60J25; 60J80

Abstract

We prove that under certain general conditions on the birth and death rates the Lebesgue-Poisson measure is a maximal irreducibility measure for the spatial birth-and-death process.

1 Introduction

The basic question of stochastic stability analysis for a Markov process is whether the chain is irreducible. The notion of irreducibility for countable state space Markov processes is not directly transferable to Markov processes with continuous state spaces. The most widely used generalization is the so called φ -irreducibility, see, e.g., Myen and Tweedie [MT93]. The aim of this paper is to prove that under certain general conditions the Lebesgue-Poisson measure is a maximal irreducible measure for continuous-space birth-and-death processes. Roughly speaking it means that, whatever the initial condition is, a set will be hit by the process with positive probability if and only if it is of positive Lebesgue-Poisson measure.

We describe and define spatial birth-and-death processes in Section 4. The pioneering works on spatial birth-and-death processes are Preston [Pre75] and Holley and Stroock [HS78]. More recent studies of various related aspects include for example [FM04, GK06, FKK12].

^{*}Email: vbezborodov@math.uni-bielefeld.de

[†]Email: luca.dipersio@univr.it

The paper is organized as follows: in Section 2 we recall the notions of ϕ -irreducibility and maximal irreducibility for measures; in Section 3 we recall the definition of the Lebesgue-Poisson measure; in Section 4 we describe the birth-and-death processes we consider and give our main result, Theorem 4.5; the proofs are collected in Section 5.

2 Irreducible and maximal irreducible measures

In what follows we shall adopt the notation used in [MT93]. Let X be a Polish space and $\mathcal{B}(X)$ be its Borel σ -algebra. We will consider a Markov chain with transition probability kernel P and initial distribution μ defined on the canonical space $\Omega = \prod_{i=0}^{\infty} X$, with Φ_n being the coordinate mappings,

$$\Phi_n((x_0, x_1, \dots)) = x_n.$$

The corresponding measure will be denoted by P_μ , so that for any Borel sets $A_0, \dots, A_n \in \mathcal{B}(\Omega)$,

$$P_\mu(\Phi_0 \in A_0, \Phi_1 \in A_1, \dots, \Phi_n \in A_n) = \int_{y_0 \in A_0} \dots \int_{y_{n-1} \in A_{n-1}} \mu(dy_0) P(y_0, dy_1) \dots P(y_{n-1}, dy_n). \quad (1)$$

Let P_x denote the distribution of Φ in Ω when the initial distribution is the Dirac measure at x , $P_x\{\Phi_0 = x\} = 1$. For any set $A \in \mathcal{B}(X)$, $\tau_A = \min\{n \geq 1 : \Phi_n \in A\}$ is called the *first return time*. Define also the return probabilities

$$L(x, A) := P_x\{\tau_A < \infty\} = P_x\{\Phi \text{ ever enters } A\}. \quad (2)$$

Definition 2.1. A finite non-trivial measure ϕ is called *ϕ -irreducible* for the chain Φ if $\phi(A) > 0$ implies that

$$L(x, A) > 0, \quad x \in X.$$

A finite non-trivial measure ψ is called *ψ -maximal irreducible* for the chain Φ if

$$(\forall x \in X : L(x, A) > 0) \Leftrightarrow \psi(A) > 0.$$

The measures ϕ and ψ from the above definition are called an *irreducibility measure* and a *maximal irreducibility measure* for Φ , respectively. The next proposition provides a sufficient condition for an irreducibility measure to be a maximal irreducibility measure.

Proposition 2.2. *If Φ is ϕ -irreducible and the measure ϕ is such that $\phi\{y : P(y, A) > 0\} = 0$ whenever $\phi(A) = 0$, then Φ is ψ -irreducible with $\psi = \phi$.*

3 Lebesgue-Poisson measure

The state space of a continuous-time, continuous-space birth and death process is

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

where $|\eta|$ is the number of points of η . $\Gamma_0(\mathbb{R}^d)$ is often called the *space of finite configurations*.

For $\eta, \zeta \in \Gamma_0$, $|\eta| = |\zeta| > 0$, we define

$$\rho(\eta, \zeta) := \min_{\varsigma} \max_{x \in \eta} \{|\varsigma(x) - x|\},$$

where minimum is taken over the set of all bijections $\varsigma : \eta \rightarrow \zeta$. For $\eta \in \Gamma_0$ and $a > 0$, let

$$\mathbf{B}_\rho(\eta, a) := \{\zeta \in \Gamma_0^{(|\eta|)} \mid \rho(\eta, \zeta) \leq a\}.$$

The σ -algebra can be defined as

$$\mathcal{B}(\Gamma_0) = \sigma(\{\emptyset\}, \mathbf{B}_\rho(\eta, a), \eta \in \Gamma_0, a > 0).$$

Let

$$\widetilde{(\mathbb{R}^d)^n} := \{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n \mid x_j \in \mathbb{R}^d, j = 1, \dots, n, x_i \neq x_j, i \neq j\}, \quad (3)$$

and let *sym* be the mapping

$$\bigsqcup_{n=0}^{\infty} \widetilde{(\mathbb{R}^d)^n} \ni (x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\} \in \Gamma_0.$$

We are now going to define the Lebesgue-Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$. For any $n \in \mathbb{N}$, let

$l_d^{\otimes n}$ be the restriction of the Lebesgue measure to $\widetilde{(\mathbb{R}^d)^n}$. We denote by $\lambda^{(n)}$ the projection of this measure on $\Gamma_0^{(n)}$,

$$\lambda^{(n)}(A) = l_d^{\otimes n}(\text{sym}^{-1}A), \quad A \in \mathcal{B}(\Gamma_0^{(n)}).$$

On $\Gamma_0^{(0)}$ the measure $\lambda^{(0)}$ is given by $\lambda^{(0)}(\{\emptyset\}) = 1$. The *Lebesgue-Poisson measure* on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined as

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}. \quad (4)$$

Let us note that the measure λ is infinite.

4 Birth-and-death processes and the main result

Denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ -algebra on \mathbb{R}^d . The evolution of a spatial birth-and-death process admits the following description. Two functions characterize the development in time, the birth rate $b : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \rightarrow [0, \infty)$ and the death rate $d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \rightarrow [0, \infty)$. If the system is in state $\eta \in \Gamma_0$ at time t , then the probability that a new particle is added (“birth” event) in a bounded set $B \in \mathcal{B}(\mathbb{R}^d)$ over the time interval $[t; t + \Delta t]$ is

$$\Delta t \int_B b(x, \eta) dx + o(\Delta t),$$

while the probability that a particle $x \in \eta$ is removed from the configuration (“death” event), over time interval $[t; t + \Delta t]$ is

$$d(x, \eta) \Delta t + o(\Delta t),$$

and simultaneous events cannot occur. In other words, the rate at which a birth occurs in B is $\int_B b(x, \eta) dx$, and the rate at which a particle $x \in \eta$ dies is $d(x, \eta)$, and no two events happen at the same time. Various aspects of birth-and-death processes are considered in, e.g., [FM04, GK06, KS06]. Here we focus our attention on the embedded Markov chain of the

birth-and-death process, namely the Markov chain on Γ_0 with transition probabilities

$$\begin{aligned} Q(\eta, \{\eta \setminus \{x\}\}) &= \frac{d(x, \eta)}{(B + D)(\eta)}, \quad x \in \eta, \quad \eta \in \Gamma_0, \\ Q(\eta, \{\eta \cup \{x\}, x \in U\}) &= \frac{\int_{x \in U} b(x, \eta) dx}{(B + D)(\eta)}, \quad U \in \mathcal{B}(\mathbb{R}^d), \eta \in \Gamma_0, \end{aligned} \quad (5)$$

where $(B + D)(\eta) = \int_{x \in \mathbb{R}^d} b(x, \eta) dx + \sum_{x \in \eta} d(x, \eta)$ is the jump rate at η .

Denote by Q_α the distribution of the Markov chain on $((\Gamma_0)^\infty, \mathcal{B}((\Gamma_0)^\infty))$ with transition probabilities (5) and initial value $\alpha \in \Gamma_0$. Here $\mathcal{B}((\Gamma_0)^\infty)$ is the σ -algebra generated by the coordinate mappings. Let $(\xi_n)_{n \in \mathbb{Z}_+}$ be the coordinate mappings $((\Gamma_0)^\infty, \mathcal{B}((\Gamma_0)^\infty))$, that is $\xi_n(\boldsymbol{\eta}) = \eta_n$ for $\boldsymbol{\eta} = (\eta_0, \eta_1, \dots) \in (\Gamma_0)^\infty$. Under Q_α , $(\xi_n)_{n \in \mathbb{Z}_+}$ is a Markov chain with transition probabilities (5).

Concerning the functions b and d , we assume that they are continuous functions in both variables, satisfying the following conditions

Condition 4.1 (Sublinear growth). There exist $c_1, c_2 > 0$ such that

$$\int_{\mathbb{R}^d} b(x, \eta) dx \leq c_1 |\eta| + c_2. \quad (6)$$

Condition 4.2. We require

$$\forall m \in \mathbb{N} : \sup_{x \in \mathbb{R}^d, |\eta| \leq m} d(x, \eta) < \infty. \quad (7)$$

Condition 4.3 (Non-degeneracy of d). The supremum

$$\inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} d(x, \eta) > 0, \quad (8)$$

Condition 4.4 (Non-degeneracy of b). For some constants $r > 0$ and $c_3 > 0$,

$$\begin{aligned} b(x, \eta) &> c_3, \text{ if there exists } y \in \eta, |x - y| \leq r, \\ \text{and } b(x, \emptyset) &> c_3 \text{ for } x \in B_\emptyset, B_\emptyset \text{ is some open ball in } \mathbb{R}^d. \end{aligned} \quad (9)$$

The following theorem constitutes the main result of the present paper.

Theorem 4.5. *The Lebesgue-Poisson measure λ is a maximal irreducibility measure for $(\xi_n)_{n \in \mathbb{N}}$.*

In other words,

$$(\forall \alpha : Q_\alpha\{(\xi_n)_{n \in \mathbb{Z}_+} \text{ ever enters } A\} > 0) \Leftrightarrow \lambda(A) > 0.$$

Remark 4.6. The second part of (9) means that points may come “out of nowhere”. We need such kind of condition in order for \emptyset not to be an absorbing state of the Markov chain $(\xi_n)_{n \in \mathbb{N}}$. Also, each of conditions (8) and (9) implies that every state $\eta \in \Gamma_0$, $\eta \neq \emptyset$, is non-absorbing.

5 Proofs

Proof of Proposition 2.2. Let ϕ be a measure satisfying conditions of the lemma. We first prove that

$$\phi\{y : L(y, A) > 0\} = 0 \text{ whenever } \phi(A) = 0. \quad (10)$$

Note that

$$\{y : L(y, A) > 0\} = \bigcup_{n \in \mathbb{N}} \{y : P^n(y, A) > 0\}. \quad (11)$$

For $A \in \mathcal{B}(X)$ and $k \in \mathbb{N}$, denote $A^{(-k)} := \{x \in X : P^k(x, A) > 0\}$. To prove (10), we will proceed by induction and show that $\phi\{y : P^n(y, A) > 0\} = 0$ as long as $\phi(A) = 0$, for all $n \in \mathbb{N}$. Assume that $\phi\{y : P^m(y, A) > 0\} = 0$ whenever $\phi(A) = 0$. Then, if $\phi(A) = 0$,

$$\begin{aligned} \phi\{y : P^{m+1}(y, A) > 0\} &= \phi\{y : \int_{x \in X} P(y, dx) P^m(x, A) > 0\} \leq \\ \phi\{y : \int_{x \in X} P(y, dx) I_{A^{(-m)}}(x) > 0\} &= \phi\{y : P(y, A^{(-m)}) > 0\} = 0. \end{aligned}$$

The base case is given in the condition, therefore (10) holds.

Assume now that the statement of the lemma does not hold, so that ϕ is not a maximal irreducible measure for Φ . Proposition 4.2.2 from [MT93] implies the existence of a maximal irreducible measure ψ' for Φ . Then there exists a set $C \in \mathcal{B}(X)$ such that $\phi(C) = 0$ whereas $\psi'(C) > 0$. By definition of irreducibility, $L(x, C) > 0$ for all $x \in X$. By (10), $\phi\{y : L(y, C) > 0\} = 0$, hence $\phi(X) = 0$, which contradicts to the non-triviality of ϕ . \square

Define a *path* of configurations as a finite sequence of configurations $\zeta_0, \zeta_1, \dots, \zeta_n$ such that $|\zeta_k \triangle \zeta_{k+1}| = 1$, $k = 0, \dots, n-1$, and if $\zeta_{k+1} = \zeta_k \cup z$, then $|z - y| \leq \frac{r}{2}$ for some $y \in \zeta_k$; that is, ζ_{k+1} is obtained from ζ_k either by adding one point to ζ_k or by removing one point from ζ_k ; in the case of the adding, it is required that the “new” point appears not further than $\frac{r}{2}$ from an “old” one. If $\zeta_k = \emptyset$, then we require $\zeta_{k+1} = \{x_\emptyset\}$, where x_\emptyset is the center of B_\emptyset . We say that such a path has length n , and we call ζ_0 and ζ_n the starting vertex and the final vertex, respectively. Also, we say that $\zeta_0, \zeta_1, \dots, \zeta_n$ is a path from ζ_0 to ζ_n .

Lemma 5.1. *For all $\eta \in \Gamma_0$ there exists a path from \emptyset to η .*

Proof. We will show that there exists a path from \emptyset to η of length less than

$$2\left(\sum_{x \in \eta} |x - x_\emptyset| \frac{4}{r} + |\eta|\right),$$

where x_\emptyset is the center of B_\emptyset .

Starting from \emptyset and only adding points, we see that there exists a path of length

$$\leq \left(\sum_{x \in \eta} |x - x_\emptyset| \frac{4}{r} + |\eta|\right),$$

with the starting vertex \emptyset and with the final vertex being some configuration $\eta' \supset \eta$. Indeed, for each $x \in \eta$ there exists a sequence of points $x_\emptyset = x_0, x_1, \dots, x_n = x$ such that $|x_i - x_{i+1}| \leq \frac{r}{4}$ and $n \leq |x - x_\emptyset| \frac{4}{r}$. Having reached $\eta' \supset \eta$, we only need to delete some points from η' . \square

Lemma 5.2. *Let $\emptyset = \eta_0, \eta_1, \dots, \eta_n$ be a path. Then for every $a > 0$*

$$Q^n(\eta_0, \mathbf{B}_\rho(\eta_n, a)) > 0.$$

Proof. Without loss of generality we can assume $a < \frac{r}{4}$. Denote $A_k = \mathbf{B}_\rho(\eta_k, a)$. We will first show that

$$\inf_{\eta \in A_k} Q(\eta, A_{k+1}) \geq \bar{c}_n \tag{12}$$

for some positive constant \bar{c}_n that depends on n but does not depend on the path we consider.

We have either $\eta_k \subset \eta_{k+1}$ or $\eta_k \supset \eta_{k+1}$. Consider first the case $\eta_k \subset \eta_{k+1}$. We know that

$\eta_{k+1} = \eta_k \cup z$, where $|z - y| \leq \frac{r}{2}$ for some $y \in \eta_k$.

Take arbitrary $\eta \in A_k$. There exists $y' \in \eta$ such that $|y - y'| \leq a$. For $x \in B_a(z)$ we have then $|x - y'| \leq |x - z| + |z - y| + |y - y'| \leq a + \frac{r}{2} + a < r$. Moreover, if $x \in B_a(z) \setminus \eta$, then $\eta \cup \{x\} \in A_{k+1}$.

From (9) we obtain

$$\begin{aligned} Q(\eta, A_{k+1}) &\geq \frac{\int_{x \in B_a(z)} b(x, \eta) dx}{(B + D)(\eta)} \geq \frac{\int_{x \in B_a(z)} c_3 dx}{(B + D)(\eta)} \\ &= \frac{c_3 a^d v_d}{(B + D)(\eta)}, \end{aligned}$$

where v_d is the volume of a unit ball in \mathbb{R}^d . By (7), the denominator of the last fraction is bounded in η , $\eta \in \bigsqcup_{k=0}^n \Gamma_0^{(k)}(\mathbb{R}^d)$. Therefore, (12) holds.

Now we turn our attention to the case when $\eta_k \supset \eta_{k+1}$. We may write $\eta_{k+1} = \eta_k \setminus \{y\}$ for some $y \in \eta_k$, and (12) follows from (8).

The statement of the lemma follows from (12), since

$$\begin{aligned} Q^n(\emptyset, \mathbf{B}_\rho(\eta_n, a)) &= \int_{\zeta_1, \zeta_2, \dots, \zeta_n} Q(\emptyset, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times \dots \times Q(\zeta_{n-1}, d\zeta_n) I_{\{\zeta_n \in \mathbf{B}_\rho(\eta_n, a)\}} \\ &\geq \int_{\zeta_1, \zeta_2, \dots, \zeta_n} Q(\emptyset, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times \dots \times Q(\zeta_{n-1}, d\zeta_n) I_{\{\zeta_k \in \mathbf{B}_\rho(\eta_k, a), k=1, \dots, n\}} \geq (\bar{c}_n)^n. \end{aligned}$$

Lemma 5.3. *Let $A \in \mathcal{B}(\Gamma_0)$, $\beta' \in \Gamma_0^{(n)}$ and $\lambda(A \cap \mathbf{B}_\rho(\beta', \frac{r}{4})) > 0$. Then*

$$Q^{2n}(\beta, A) > 0$$

for any $\beta \in \mathbf{B}_\rho(\beta', \frac{r}{4})$.

The idea of the proof. Let $\beta = \{x_1, \dots, x_n\}$. The event R described in the next sentence has positive probability. Let $\xi_1 = \beta \cup y_1$ for some $y_1 \in B_{\frac{r}{4}}(x_1)$, $\xi_2 = \xi_1 \setminus x_1$, $\xi_3 = \xi_2 \cup y_2$ for some $y_2 \in B_{\frac{r}{4}}(x_2)$, $\xi_4 = \xi_3 \setminus x_2$, and so on, so that $\xi_{2n} = \xi_{2n-1} \setminus x_n$. We will see that $Q_\beta\{\xi_{2n} \in A \mid R\} > 0$.

Proof.

Fix $\beta = \{x_1, \dots, x_n\}$. Consider a measurable subset Ξ of $(\Gamma_0)^{(2n)}$,

$$\Xi = \left\{ (\zeta_1, \dots, \zeta_{2n}) \mid \zeta_{2k-1} = \{y_1, \dots, y_k, x_k, \dots, x_n\}, \zeta_{2k} = \{y_1, \dots, y_k, x_{k+1}, \dots, x_n\}, k = 1, \dots, n, \right. \\ \left. \text{for some distinct } y_1, \dots, y_n \in \mathbb{R}^d \text{ satisfying } |y_k - x_k| \leq \frac{r}{4} \right\}.$$

Define $R = \{(\xi_1, \dots, \xi_{2n}) \in \Xi\}$.

By the Markov property,

$$\begin{aligned} Q^{2n}(\beta, A) &= \int_{\zeta_1, \zeta_2, \dots, \zeta_{2n}} Q(\beta, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times \dots \times Q(\zeta_{2n-1}, d\zeta_{2n}) I_{\{\xi_{2n} \in A\}} \\ &\geq \int_{\zeta_1, \zeta_2, \dots, \zeta_{2n}} Q(\beta, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times \dots \\ &\quad \times Q(\zeta_{2n-1}, d\zeta_{2n}) I_{\{(\zeta_1, \dots, \zeta_{2n}) \in \Xi\}} I_{\{(\zeta_2 \setminus \zeta_1) \curlyvee (\zeta_4 \setminus \zeta_3) \curlyvee \dots \curlyvee (\zeta_{2n} \setminus \zeta_{2n-1}) \in \text{sym}^{-1} A\}}. \end{aligned} \quad (13)$$

Here for singletons $\mathbb{S}_1 = \{s_1\}, \mathbb{S}_2 = \{s_2\}, \dots, \mathbb{S}_n = \{s_n\}$ we define

$$\mathbb{S}_1 \curlyvee \mathbb{S}_2 \curlyvee \dots \curlyvee \mathbb{S}_n = (s_1, s_2, \dots, s_n).$$

Note that $\zeta_{2n} = (\zeta_2 \setminus \zeta_1) \curlyvee (\zeta_4 \setminus \zeta_3) \curlyvee \dots \curlyvee (\zeta_{2n} \setminus \zeta_{2n-1})$ if $(\zeta_1, \dots, \zeta_{2n}) \in \Xi$.

From the definition of the Lebesgue Poisson measure we have

$$l(\text{sym}^{-1} A) = n! \lambda(A), \quad (14)$$

where l_d^n is the Lebesgue measure on $(\mathbb{R}^d)^n$.

Define a measure σ on $(\prod_{k=1}^n B_{\frac{r}{4}}(x_k), \mathcal{B}(\prod_{k=1}^n B_{\frac{r}{4}}(x_k)))$ by

$$\begin{aligned} \sigma(D) &= \int_{\zeta_1, \zeta_2, \dots, \zeta_{2n}} Q(\beta, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times \dots \times Q(\zeta_{2n-1}, d\zeta_{2n}) \\ &\quad \times I_{\{(\zeta_1, \dots, \zeta_{2n}) \in \Xi\}} I_{\{(\zeta_2 \setminus \zeta_1) \curlyvee (\zeta_4 \setminus \zeta_3) \curlyvee \dots \curlyvee (\zeta_{2n} \setminus \zeta_{2n-1}) \in D\}}, \quad D \in \mathcal{B}\left(\prod_{k=1}^n B_{\frac{r}{4}}(x_k)\right). \end{aligned}$$

We can rewrite (13) as

$$Q^{2n}(\beta, A) \geq \sigma(\text{sym}^{-1} A). \quad (15)$$

We will show that

$$\sigma(D) \geq \tilde{c}_3 l_d^n(D), \quad D \in \mathcal{B}\left(\prod_{k=1}^n B_{\frac{r}{4}}(x_k)\right) \quad (16)$$

for some constant $\tilde{c}_3 > 0$.

The statement of the lemma is a consequence of (14), (15) and (16). To establish (16) we only need to consider sets of the form $D_1 \times \dots \times D_n$, $D_j \in \mathcal{B}(B_{\frac{r}{4}}(x_j))$. Define

$$\Xi_{(D_1, \dots, D_n)} = \left\{ (\zeta_1, \dots, \zeta_{2n}) \mid \zeta_{2k-1} = \{y_1, \dots, y_k, x_k, \dots, x_n\}, \zeta_{2k} = \{y_1, \dots, y_k, x_{k+1}, \dots, x_n\}, k = 1, \dots, n, \right. \\ \left. \text{for some distinct } y_k \in D_k \right\}.$$

We have

$$\sigma(D_1 \times \dots \times D_n) = \int_{\zeta_1, \zeta_2, \dots, \zeta_{2n}} Q(\beta, d\zeta_1) Q(\zeta_1, d\zeta_2) Q(\zeta_2, d\zeta_3) \times \dots \\ \times Q(\zeta_{2n-1}, d\zeta_{2n}) I\{(\zeta_1, \dots, \zeta_{2n}) \in \Xi_{(D_1, \dots, D_n)}\}.$$

Fix $z_j \in D_j$. Using our assumptions on b and d , we see that

$$Q\left(\{z_1, \dots, z_k, x_k, \dots, x_n\}, \{\{z_1, \dots, z_k, x_{k+1}, \dots, x_n\}\}\right) = \frac{d(x_k, \{z_1, \dots, z_k, x_k, \dots, x_n\})}{(B+D)\{z_1, \dots, z_k, x_k, \dots, x_n\}} \\ \geq \frac{d(x_k, \{z_1, \dots, z_k, x_k, \dots, x_n\})}{\sup\{(B+D)(\eta) \mid |\eta| \leq n+1\}} \geq \frac{\inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} d(x, \eta)}{\sup\{(B+D)(\eta) \mid |\eta| \leq n+1\}},$$

and

$$Q\left(\{z_1, \dots, z_k, x_{k+1}, \dots, x_n\}, \{\{z_1, \dots, z_k, y_{k+1}, x_{k+1}, \dots, x_n\} \mid y_{k+1} \in D_{k+1}\}\right) = \\ = \frac{\int_{y \in D_{k+1}} b(y, \{z_1, \dots, z_k, x_{k+1}, \dots, x_n\}) dy}{(B+D)(\{z_1, \dots, z_k, x_{k+1}, \dots, x_n\})} \geq \frac{c_3 l_d(D_{k+1})}{(B+D)(\{z_1, \dots, z_k, x_{k+1}, \dots, x_n\})},$$

where l_d is the Lebesgue measure on \mathbb{R}^d . Hence

$$\sigma(D_1 \times \dots \times D_n) \geq \left(\frac{\inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} d(x, \eta)}{\sup\{(B+D)(\eta) : |\eta| \leq n+1\}} \right)^n \prod_{j=1}^n \frac{c_3 l_d(D_j)}{\sup\{(B+D)(\eta) : |\eta| \leq n+1\}}.$$

It remains to note that $\prod_{j=1}^n l_d(D_j) = l_d^n(D_1 \times \dots \times D_n)$.

Proof of Theorem 4.5. We will first establish ϕ -irreducibility. Starting from any configuration, the process may go extinct in finite time: for all $\eta \in \Gamma_0(\mathbb{R}^d)$

$$Q_\eta\{\xi_k = \emptyset \text{ for some } k > 0\} > 0.$$

Therefore, it is sufficient to show that

$$L(\emptyset, A) > 0 \quad \text{whenever } \lambda(A) > 0, A \in \mathcal{B}(\Gamma_0). \quad (17)$$

Let us take $A \in \mathcal{B}(\Gamma_0)$ with $\lambda(A) > 0$. There exists $n \in \mathbb{N}$ and $\beta' \in \Gamma_0^{(n)}$ such that

$$\lambda(A \cap \mathbf{B}_\rho(\beta', \frac{r}{4})) > 0. \quad (18)$$

By Lemma 5.1 there exists a path from \emptyset to β' . Denote by m the length of this path. Applying Lemma 5.2 and Lemma 5.3 we get

$$Q^{m+2n}(\emptyset, A) \geq \int_{\beta \in \mathbf{B}_\rho(\beta', \frac{r}{4})} Q^m(\emptyset, d\beta) Q^{2n}(\beta, A) > 0,$$

which proves (17).

Now let us prove that λ is a maximal irreducibility measure for $(\xi_n)_{n \in \mathbb{N}}$. Taking into account Proposition 2.2, we see that it suffices to show that for all $A \subset \Gamma_0(\mathbb{R}^d)$ with $\lambda(A) = 0$ we have

$$\lambda\{\eta : Q(\eta, A) > 0\} = 0. \quad (19)$$

With no loss of generality, we assume that $A \subset \Gamma_0^{(n)}(\mathbb{R}^d)$, $n \geq 2$. We have $\text{sym}^{-1}(A) \subset (\mathbb{R}^d)^n$ and $l_d^n(\text{sym}^{-1}(A)) = 0$. Now, $\eta \in \Gamma_0^{(n+1)}(\mathbb{R}^d)$ and $Q(\eta, A) > 0$ if and only if η may be represented as $\xi \cup \{x\}$, where $\xi \in A$, $x \in \mathbb{R}^d \setminus \xi$. Then we also have for any $y = (y_1, \dots, y_{n+1}) \in \text{sym}^{-1}(\eta)$

$$\check{\Pi}_j y \in \text{sym}^{-1}(A)$$

for some $j \in \{1, 2, \dots, n+1\}$, where $\check{\Pi}_j y = (y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{n+1}) \in (\mathbb{R}^d)^n$.

Since $l_d^n(\text{sym}^{-1}(A)) = 0$, we also have $l_d^{(n+1)}(\check{\Pi}(\cdot)_j^{-1}(\text{sym}^{-1}(A))) = 0$, and consequently

$$\lambda\{\eta : \eta \in \Gamma_0^{(n+1)}, Q(\eta, A) > 0\} = 0. \quad (20)$$

Similarly, if $\eta \in \Gamma_0^{(n-1)}(\mathbb{R}^d)$ and $Q(\eta, A) > 0$, then for $y \in \text{sym}^{-1}(\eta)$

$$l_d\{z \in \mathbb{R}^d : (z, y) \in \text{sym}^{-1}(A)\} > 0. \quad (21)$$

because a “newly born” point has an absolutely continuous distribution with respect to the Lebesgue measure on \mathbb{R}^d , in the sense that $Q(\eta, \{\eta \cup z \mid z \in D\}) = 0$ if $l_d(D) = 0$. However, the set of all y satisfying (21) has zero Lebesgue measure, otherwise we would have

$$l_d^n(\text{sym}^{-1}(A)) = \int l_d^{(n-1)}(dy) l_d\{z : (z, y) \in \text{sym}^{-1}(A)\} > 0.$$

Therefore,

$$\lambda\{\eta : \eta \in \Gamma_0^{(n-1)}, Q(\eta, A) > 0\} = 0. \quad (22)$$

Note that in cases $n = 0, 1$ some changes should be made in the proofs of (20), (22), because of the special structure of $\Gamma_0^{(0)}(\mathbb{R}^d) = \{\emptyset\}$. Now, we also have

$$\{\eta : \eta \in \Gamma_0^{(k)}, P(\eta, A) > 0\} = \emptyset,$$

$k \neq n - 1, n + 1$, $n \geq 0$. Consequently, (20) and (22) imply (19). \square

Acknowledgement

Viktor Bezborodov is supported by the Department of Computer Science at the University of Verona.

References

- [FKK12] D. Finkelshtein, O. Kutovyi, and Y. G. Kondratiev. Semigroup approach to birth-and-death stochastic dynamics in continuum. *Journal of Functional Analysis*, 262(3):12741308, 2012.

- [FM04] N. Fournier and S. Méléard. A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Probab.*, 14(4):1880–1919, 2004.
- [GK06] N. L. Garcia and T. G. Kurtz. Spatial birth and death processes as solutions of stochastic equations. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1:281–303, 2006.
- [HS78] R. A. Holley and D. W. Stroock. Nearest neighbor birth and death processes on the real line. *Acta Math*, 140(1-2):103154, 1978.
- [KS06] Y. G. Kondratiev and A. V Skorokhod. On contact processes in continuum. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 9(2):187198, 2006.
- [MT93] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Springer, 1993.
- [Pre75] C. Preston. Spatial birth-and-death processes. In *Proceedings of the 40th Session of the International Statistical Institute*, volume 46 of *Bull. Inst. Internat. Statist*, pages 371391, 405408, 1975.